

A ZERO DENSITY RESULT FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. In this article, we prove an explicit bound for $N(\sigma, T)$, the number of zeros of the Riemann zeta function satisfying $\Re s \geq \sigma$ and $0 \leq \Im s \leq T$. This result provides a significant improvement to Rosser's bound for $N(T)$ when used for estimating prime counting functions.

1. INTRODUCTION

In recent years, it has become apparent that explicit results concerning prime numbers are required to solve important problems in number theory. In particular, the impressive works of Ramaré [18], Tao [30], and Helfgott [13] related to Goldbach's conjecture highlight the need of better explicit bounds for finite sums over primes. For instance, they make use of [4], [21], [22], [24], [25], [26], [28]. Moreover articles of Rosser and Schoenfeld ([24], [25], [26], [27], [28]), Dusart ([5], [6], [7], [8]), and Ramaré and Rumely [23] are extensively used in a wide range of fields including Diophantine approximation, cryptography, and computer science. These results on primes rely heavily on explicit estimates of sums over the non-trivial zeros of the Riemann zeta function. More precisely, they rely on three key ingredients: a numerical verification of the Riemann Hypothesis (RH), an explicit zero-free region, and explicit bounds for the number of zeros in the critical strip up to a fixed height T .

In 1986, van de Lune et al. [34] established that RH had been verified for all zeros ϱ verifying $|\Im \varrho| \leq H_0$ with $H_0 = 545\,439\,823$. In 2011, Platt [15] [16] proved that $H_0 = 30\,610\,046\,000$ is admissible. Previously, Wedeniowski [35] in 2001 and Gourdon [11] in 2004 had announced higher values for H_0 . As Platt's computations are more rigorous (he employs interval arithmetic), we decide to use his value throughout this article:

$$H_0 = 3.061 \cdot 10^{10}.$$

For the latest explicit results about zero-free regions for the Riemann zeta function, we refer the reader to [14] and [10].

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Let $\sigma \geq 0.55$. We consider $N(\sigma, T)$, the number of zeros of the Riemann zeta function in the region $\sigma \leq \Re s \leq 1$ and $0 \leq \Im s \leq T$. Trivially we have that $N(\sigma, T) = 0$ for all $T \leq H_0$. We prove here an explicit bound for $N(\sigma, T)$ valid in the range $T \geq H_0$.

Theorem 1.1. *Let $\sigma \geq 0.55$ and $T \geq H_0$. Let σ_0 and H such that $0.5208 < \sigma_0 < 0.9723$, $\sigma_0 < \sigma$, and $10^3 \leq H \leq H_0$. Then there exist b_1, b_2, b_3 , positive constants depending on σ, σ_0, H , such that:*

$$N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3.$$

The b_i 's are defined in (6.3).

We rewrite this as $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$, for $T \geq H_0$. Numerical values of the b_i 's and c_i 's are recorded at the end of this article in Table 1. For example, for $\sigma \geq 17/20$ and $T \geq H_0$, we have

$$N(\sigma, T) \leq 0.5561T + 0.7586 \log T - 268\,658.$$

Let $N(T)$ be the number of non-trivial zeros ρ with imaginary part $0 \leq \Im \rho \leq T$. We recall that Rosser [24] proved

$$(1.1) \quad \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq a \log T + b \log \log T + c,$$

with $a = 0.137$, $b = 0.443$, $c = 1.588$. Note that Rosser's result got recently improved by Trudgian [33, Corollary 1] with $a = 0.111$, $b = 0.275$, $c = 2.450$. A trivial bound for $N(\sigma, T)$ follows from the inequalities $N(\sigma, T) \leq \frac{1}{2}N(T)$ and (1.1):

$$N(\sigma, T) \leq \frac{T}{4\pi} \log \left(\frac{T}{2\pi e} \right) (1 + o(1)).$$

Note that when T is asymptotically large, then a factor of $\log T$ is saved. Moreover, we have $c_1 \sim \frac{\log \zeta(2\sigma_0)}{4\pi(\sigma - \sigma_0)}$ where σ_0 is a parameter which value can be chosen to make c_1 as small as possible. Another feature of Theorem 1.1 is the factor $T - H$: when T is near H_0 , we choose H to be close to H_0 so as to make $N(\sigma, T)$ of size $\log H_0$. This saves a factor of size H_0 . As an example, for $\sigma \geq 17/20$ and $T = H_0 + 1$, we choose $H = H_0 - 1$ and σ_0 as in Table 1 and obtain $N(\sigma, H_0 + 1) \leq 156$ while (1.1) gives $5.2 \cdot 10^{10}$ (with either Rosser's or Trudgian's values).

The key motivation for establishing Theorem 1.1 is to use it in place of (1.1) and thus to provide improved explicit bounds for Chebyshev's prime counting functions. We prove in [9] that, for all $x \geq e^b$,

$$(1.2) \quad |\psi(x) - x| \leq \epsilon_b x,$$

where b is a fixed positive constant, and ϵ_b is an effective positive constant. For example, for $x \geq e^{50}$ we obtain $\epsilon_{50} = 9.461 \cdot 10^{-10}$ while Dusart [7, Theorem 2] obtained $0.905 \cdot 10^{-7}$.

Despite a very rich history of asymptotic results, there were almost no explicit bounds for $N(\sigma, T)$. Ramaré proved in an unpublished manuscript [19] that, for $T \geq 2000$, $Q \geq 10$, and $T \geq Q$,

$$\sum_{q \leq Q} \sum_{\chi \bmod *q} N(\sigma, T, \chi) \leq 157(Q^5 T^3)^{1-\sigma} \log^{4-\sigma}(Q^2 T) + 6Q^2 \log^2(Q^2 T),$$

where $\sum_{\chi \bmod *q}$ denotes the sum over primitive Dirichlet characters χ to the modulus q , and $N(\sigma, T, \chi)$ counts the number of zeros ρ of the Dirichlet L -function $L(s, \chi)$ satisfying $\sigma < \Re \rho < 1$ and $0 < \Im \rho < T$. Taking $Q = 10$ and restricting the left sum to $q = 1$, it follows that

$$(1.3) \quad N(\sigma, T) \leq 157(100\,000T^3)^{1-\sigma} \log^{4-\sigma}(100T) + 600 \log^2(100T).$$

Our main theorem improves Ramaré's result for certain values of σ and T : he obtains $N(17/20, 10 \cdot H_0) \leq 2.675 \cdot 10^{12}$ while we have $N(17/20, 10 \cdot H_0) \leq 3.404 \cdot 10^{10}$. In 2010, Cheng [3] obtained the weaker result:

$$(1.4) \quad N(\sigma, T) \leq 453\,472.54 T^{8/3(1-\sigma)} (\log T)^5,$$

for all $\sigma \geq 5/8$ and $T \geq \exp(\exp(18)) \simeq 10^{28\,515\,762}$. His method is based on Ford's [10] effective version of Korobov-Vinogradov's bound for the Riemann zeta function. He applied (1.4) to deduce explicit results on primes between consecutive cubes. Note that Cheng's result is not valid in the region $T \leq \exp(\exp(18))$ while most applications require bounds for T as small as H_0 .

In order to prove Theorem 1.1 we establish two intermediate theorems about $\zeta(s)$ in the critical strip: an effective version of a Dirichlet polynomial approximation, and an explicit estimate for the second moment.

Theorem 1.2. *Let $t_0 > 0$, $s = \sigma + it$ with $\sigma \geq 1/2$, $t \geq t_0$ and $c > \frac{1}{2\pi}$. Then*

$$\zeta(s) = \sum_{1 \leq n < ct} \frac{1}{n^s} + R(s)$$

with $|R(s)| \leq C(\sigma, c)t^{-\sigma}$, and

$$(1.5) \quad C(\sigma, c) = \left(c + \frac{1}{2} + \frac{3\sqrt{1+1/t_0^2}}{2\pi} \left(\frac{\zeta(2)}{2\pi c} + 1 + \frac{1}{2\pi c - 1} \right) \right) c^{-\sigma}.$$

We apply the theorem for $c = 1$ and for t_0 the height of the first zero of zeta.

Corollary 1.3. *Let $\sigma \geq 1/2$ and $t \geq 14.1347$. Then*

$$(1.6) \quad \left| \zeta(s) - \sum_{1 \leq n < t} \frac{1}{n^s} \right| \leq c_0 t^{-\sigma}, \quad \text{where } c_0 = 2.1946.$$

This is to compare to Proposition 1 of Cheng [2] who obtained 5.505 instead of $2.1946t^{-\sigma}$. When $\sigma \geq 1/2$ and $0 \leq t \leq 15$, a Mathematica computation gives us that $|\zeta(s) - \sum_{1 \leq n < t} n^{-s}| \leq 43t^{-\sigma}$.

Theorem 1.4. *Let $0.5208 < \sigma_0 < 0.9723$ and $10^3 \leq H \leq H_0$. We define*

$$(1.7) \quad \begin{aligned} \epsilon_1(\sigma_0, H) = & \frac{4H_0}{H_0 - H} \left(\frac{(\log H_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{(2\sigma_0 - 1) \log H_0}{2(1-\sigma_0)H_0} \right. \\ & + \frac{\max \left(0, \frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2} \right)}{H_0} + \frac{(2-\sigma_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)^2} \\ & \left. - \frac{\sigma_0 H_0^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{H_0^{-2\sigma_0}}{2(2\sigma_0 - 1)} + \frac{H_0^{-2\sigma_0-1}}{2} \right), \end{aligned}$$

$$(1.8) \quad \epsilon_2(\sigma_0, H) = \frac{c_0^2}{2\sigma_0 - 1} \frac{H^{-(2\sigma_0-1)} - H_0^{-(2\sigma_0-1)}}{H_0 - H},$$

$$(1.9) \quad \epsilon_3(\sigma_0, H) = 2\sqrt{\epsilon_2(\sigma_0, H)(\zeta(2\sigma_0) + \epsilon_1(\sigma_0, H))},$$

$$(1.10) \quad \mathcal{E}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3.$$

Then, for all $T \geq H_0$, we have

$$\begin{aligned} & \frac{1}{T-H} \int_H^T |\zeta(\sigma_0 + it)|^2 dt \leq \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H), \\ \text{and } & \int_H^T \log |\zeta(\sigma_0 + it)| dt \leq \frac{T-H}{2} \log \left(\zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right). \end{aligned}$$

For the rest of this article H , T , σ_0 , and σ satisfy

$$(1.11) \quad H_0 = 3.061 \cdot 10^{10}, 10^3 \leq H \leq H_0 \leq T, 0.5208 < \sigma_0 < 0.9723, \sigma_1 = 1.5002, \sigma_0 < \sigma < \sigma_1.$$

2. APPROXIMATE FORMULA FOR $\zeta(\sigma + it)$ - PROOF OF THEOREM 1.2

Let $s = \sigma + it$ with $1/2 < \sigma < 1$ and $t \geq 2$. Let $x = ct$ with $c > \frac{1}{2\pi}$, and let N be a positive integer. Theorem 1.2 gives an explicit version of an approximation formula for zeta, as proven by Hardy and Littlewood in [12].

Proof. We start with the classical identity [31, equation 3.5.3]

$$(2.1) \quad \zeta(s) - \sum_{1 \leq n < x} \frac{1}{n^s} = \sum_{x \leq n \leq N} \frac{1}{n^s} + s \int_N^\infty \frac{((u))}{u^{s+1}} du - \frac{N^{1-s}}{1-s} - \frac{1}{2} N^{-s},$$

where $((u)) = [u] - u + 1/2$. The summation formula [31, equation 2.1.2] gives

$$\sum_{x \leq n < N} \frac{1}{n^s} = \int_x^N \frac{du}{u^s} - \frac{((x))}{x^s} + s \int_x^N \frac{((u))}{u^{1+s}} du = \frac{N^{1-s} - x^{1-s}}{1-s} - \frac{((x))}{x^s} + s \int_x^N \frac{((u))}{u^{1+s}} du.$$

We have the bounds

$$\left| \frac{x^{1-s}}{1-s} \right| \leq \frac{x^{1-\sigma}}{t}, \quad \left| \frac{((x))}{x^s} \right| \leq \frac{x^{-\sigma}}{2}, \quad \left| s \int_N^\infty \frac{((u))}{u^{s+1}} du \right| \leq \frac{|s|}{2} \int_N^\infty \frac{1}{u^{\sigma+1}} du = \frac{|s|}{2\sigma N^\sigma}.$$

Thus

$$(2.2) \quad \left| \zeta(s) - \sum_{1 \leq n < x} \frac{1}{n^s} \right| \leq x^{1-\sigma} t^{-1} + \frac{x^{-\sigma}}{2} + \left| s \int_x^N \frac{((u))}{u^{1+s}} du \right| + \frac{|s|}{2\sigma} N^{-\sigma} + \frac{1}{2} N^{-\sigma}.$$

The choice $x = ct$ is made to balance the error term $x^{1-\sigma} t^{-1} + \frac{x^{-\sigma}}{2}$. We appeal to the Fourier series of $((x))$ to obtain a smaller bound for the integral expression. For $u \notin \mathbb{N}$, we have [31, p. 74]

$$((u)) = [u] - u + 1/2 = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin(2\pi\nu u)}{\nu}.$$

Lebesgue's bounded convergence theorem applies, and we can exchange the order of the integral and the summation. We obtain

$$(2.3) \quad \int_x^N \frac{((u))}{u^{1+s}} du = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_x^N \frac{\sin(2\pi\nu u)}{u^{1+s}} du = \sum_{\nu=1}^{\infty} \frac{I(\nu) - I(-\nu)}{\nu},$$

where the integral I is given by

$$(2.4) \quad I(h) = \frac{1}{2\pi i} \int_x^N \frac{e^{2i\pi(hu - \frac{t \log u}{2\pi})}}{u^{\sigma+1}} du = \frac{1}{2\pi} \int_x^N F(h, u) d(e^{2\pi i(f(u) + hu)})$$

with $F(h, u) = \frac{u^{-\sigma}}{t - 2\pi u h}$ and $f(u) = -\frac{t \log u}{2\pi}$. Since

$$\frac{\partial}{\partial u} F(h, u) = u^{-\sigma} \frac{-\sigma t u^{-1} + 2\pi h(\sigma + 1)}{(t - 2\pi u h)^2},$$

it is easy to check that $F(-\nu, u)$ is positive and decreases with u , and that $F(\nu, u)$ is negative and increases with u .

We now apply the second mean value theorem from [32, section 12.3]:

Lemma 2.1. *If $j(x)$ is integrable over (a, b) , and $\phi(x)$ is positive, bounded, and non-increasing, then there exists $\xi \in (a, b)$ such that*

$$\int_a^b \phi(x) j(x) dx = \phi(a+0) \int_a^\xi j(x) dx.$$

First, we consider $I(-\nu)$. We separate the real and imaginary part in $d(e^{2\pi i(f(u) + hu)})$ in (2.4) and we apply the Lemma for $\phi(u) = F(-\nu, u)$. We

consider $j(u)du = d(\cos(2\pi(f(u) - \nu u)))$, and $j(u)du = d(\sin(2\pi(f(u) - \nu u)))$ respectively. We obtain that there exist $\xi_1, \xi_2 \in (x, N)$ such that

$$\begin{aligned} 2\pi I(-\nu) &= F(-\nu, x) \cos(2\pi(f(\xi_1) - \nu \xi_1)) - F(-\nu, x) e^{2\pi i(f(x) - \nu x)} \\ &\quad + iF(-\nu, x) \sin(2\pi(f(\xi_2) - \nu \xi_2)). \end{aligned}$$

It follows that

$$(2.5) \quad |I(-\nu)| \leq \frac{3}{2\pi} F(-\nu, x) = \frac{3}{2\pi} \frac{(ct)^{-\sigma}}{t + 2\pi ct\nu} \leq \frac{3}{(2\pi)^2} \frac{c^{-\sigma-1} t^{-\sigma-1}}{\nu}.$$

A similar argument applies to $I(\nu)$. We obtain

$$(2.6) \quad |I(\nu)| \leq -\frac{3}{2\pi} F(\nu, ct) = \frac{3}{2\pi} \frac{(ct)^{-\sigma}}{2\pi ct\nu - t} \leq \begin{cases} \frac{3}{2\pi} \frac{c^{-\sigma} t^{-\sigma-1}}{\nu-1} & \text{if } \nu \geq 2, \\ \frac{3}{2\pi} \frac{c^{-\sigma} t^{-\sigma-1}}{2\pi c-1} & \text{if } \nu = 1. \end{cases}$$

Using the simplification $\sum_{\nu=2}^{\infty} \frac{1}{\nu(\nu-1)} = 1$, $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \zeta(2)$, and $\frac{|s|}{t} \leq \sqrt{1 + 1/t^2}$, we put together (2.3), (2.5), and (2.6), and obtain the bound

$$\begin{aligned} \left| s \int_x^N \frac{((u))}{u^{1+s}} du \right| &\leq |s| \sum_{\nu=1}^{\infty} \frac{|I(\nu)| + |I(-\nu)|}{\nu} \\ &\leq \frac{3\sqrt{1 + 1/t^2}}{2\pi} \left(1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c} \right) c^{-\sigma} t^{-\sigma}. \end{aligned}$$

Letting $N \rightarrow \infty$, inequality (2.2) becomes

$$\left| \zeta(s) - \sum_{1 \leq n < ct} \frac{1}{n^s} \right| \leq \left(c + \frac{1}{2} + \frac{3\sqrt{1 + 1/t^2}}{2\pi} \left(1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c} \right) \right) (ct)^{-\sigma}.$$

□

Remark 2.2. A careful reading of Cheng's proof shows that his error term has size $\mathcal{O}(t^{1-2\sigma})$, instead of our $\mathcal{O}(t^{-\sigma})$. This comes from the fact that he bounds directly the terms $\frac{N^{1-s}}{1-s}$, instead of eliminating them as we did.

3. EXPLICIT UPPER BOUND FOR THE SECOND MOMENT OF ZETA - PROOF OF THEOREM 1.4

We recall that σ_0, T, H are as in (1.11). By Theorem 1.2, we have the identity

$$(3.1) \quad \frac{1}{T-H} \int_H^T |\zeta(\sigma_0 + it)|^2 dt = D(\sigma_0, T, H) + E_1(\sigma_0, T, H) + E_2(\sigma_0, T, H) + E_3(\sigma_0, T, H),$$

where

$$\begin{aligned}
D(\sigma_0, T, H) &= \frac{1}{T-H} \int_H^T \sum_{1 \leq n < t} \frac{1}{n^{2\sigma_0}} dt, \\
E_1(\sigma_0, T, H) &= \frac{2}{T-H} \int_H^T \sum_{1 \leq n < m < t} \frac{\cos(t \log(m/n))}{(nm)^{\sigma_0}} dt, \\
E_2(\sigma_0, T, H) &= \frac{1}{T-H} \int_H^T |R(\sigma_0 + it)|^2 dt, \\
E_3(\sigma_0, T, H) &= \frac{2}{T-H} \Re \int_H^T \sum_{1 \leq n < t} \frac{R(\sigma_0 + it)}{n^{\sigma_0 + it}} dt.
\end{aligned}$$

We recall here some basic inequalities that we use throughout the following argument. Let $A, B \in \mathbb{N}$. If f is decreasing and positive, then

$$(3.2) \quad \sum_{A \leq j \leq B} f(j) \leq f(A) + \int_A^B f(u) du.$$

For $\sigma_0 > 1/2$, we bound trivially the diagonal term:

$$(3.3) \quad D(\sigma_0, T, H) \leq \zeta(2\sigma_0).$$

We interchange summation order in the off-diagonal terms $E_1(\sigma_0, T, H)$ and use the fact that $\int_u^v \cos(at) dt \leq \frac{2}{a}$ when $a \neq 0$:

$$E_1(\sigma_0, T, H) \leq \frac{4}{T-H} \sum_{1 \leq n < m < T} \frac{(nm)^{-\sigma_0}}{\log(m/n)}.$$

We use the fact that, for $\lambda > 1$ and $\sigma < 1$, $\frac{1}{\log \lambda} \leq 1 + \frac{\lambda^{1-\sigma}}{\lambda-1}$. Taking $\lambda = \frac{m}{n}$, we obtain

$$(3.4) \quad E_1(\sigma_0, T, H) \leq \frac{4}{T-H} \sum_{1 \leq n < m < T} (nm)^{-\sigma_0} + \frac{4}{T-H} \sum_{1 \leq n < m < T} \frac{m^{1-2\sigma_0}}{m-n}.$$

For the first sum, we complete the square

$$\sum_{1 \leq n < m < T} (nm)^{-\sigma_0} = \frac{1}{2} \left(\sum_{k < T} k^{-\sigma_0} \right)^2 - \frac{1}{2} \sum_{k < T} k^{-2\sigma_0} = \frac{1}{2} \left(\sum_{k < T} k^{-\sigma_0} \right)^2 - \frac{1}{2} \left(\zeta(2\sigma_0) - \sum_{k \geq T} k^{-2\sigma_0} \right),$$

and use (3.2) with $f(t) = t^{-\sigma_0}$ and $f(t) = t^{-2\sigma_0}$ to bound the resulting sums. We obtain

$$(3.5) \quad \sum_{1 \leq n < m < T} (nm)^{-\sigma_0} \leq \frac{T^{2(1-\sigma_0)}}{2(1-\sigma_0)^2} - \frac{\sigma_0 T^{1-\sigma_0}}{(1-\sigma_0)^2} + \frac{\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2} \zeta(2\sigma_0) - \frac{T^{1-2\sigma_0}}{2(1-2\sigma_0)} + \frac{T^{-2\sigma_0}}{2}.$$

We consider $k = m - n$ and separate variables in the second sum of (3.4) and use (3.2), with $f(t) = t^{1-2\sigma_0}$ and $f(t) = t^{-1}$, to bound the resulting

sums:

$$(3.6) \quad \sum_{1 \leq n < m < T} \frac{m^{1-2\sigma_0}}{m-n} \leq \left(\sum_{1 \leq m < T} m^{1-2\sigma_0} \right) \left(\sum_{1 \leq k < T} k^{-1} \right) \\ \leq \frac{(\log T)T^{2(1-\sigma_0)}}{2(1-\sigma_0)} + \frac{T^{2(1-\sigma_0)}}{2(1-\sigma_0)} + \log T + 1 - \frac{1}{2(1-\sigma_0)} - \frac{\log T}{2(1-\sigma_0)}.$$

Together with (3.4), (3.5) and (3.6), we obtain

$$E_1(\sigma_0, T, H) \leq \frac{4T}{T-H} \left(\frac{(\log T)T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)} \frac{\log T}{T} + \left(\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0) \right) \frac{1}{T} \right) \\ + \frac{2-\sigma_0}{2(1-\sigma_0)^2} T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{T^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{1}{2} T^{-2\sigma_0-1}.$$

We denote

$$E_{11}(\sigma_0, T) = \frac{(\log T)T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)} \frac{\log T}{T}, \\ E_{12}(\sigma_0, T) = \left(\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0) \right) \frac{1}{T}, \\ E_{13}(\sigma_0, T) = \frac{2-\sigma_0}{2(1-\sigma_0)^2} T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2}, \\ E_{14}(\sigma_0, T) = \frac{T^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{1}{2} T^{-2\sigma_0-1},$$

and we now study their behavior with respect to $T \geq H_0$. It is immediate that E_{14} decreases with T . Considering the fact that $\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0)$ changes sign at $\sigma_0 = 0.679785\dots$, we obtain

$$E_{12}(\sigma_0, T) \leq \max(0, E_{12}(\sigma_0, H_0)).$$

For $0.5208 < \sigma_0 < 1$, we find

$$\frac{\partial E_{11}(\sigma_0, T)}{\partial T} = \frac{-(T^{2(1-\sigma_0)} - 1)((2\sigma_0-1)(\log T) - 1) + 2(1-\sigma_0)}{2(1-\sigma_0)T^2} \leq 0,$$

and, when $\sigma_0 \leq 0.9723$, that

$$\frac{\partial E_{13}(\sigma_0, T)}{\partial T} = \left(-\frac{(2-\sigma_0)(2\sigma_0-1)}{2} T^{1-\sigma_0} + \sigma_0^2 \right) \frac{T^{-1-\sigma_0}}{(1-\sigma_0)^2} \leq 0.$$

Thus $E_{11}(\sigma_0, T)$ and $E_{13}(\sigma_0, T)$ decrease with $T \geq H_0$. We conclude that, for $T \geq H_0$ and $0.5208 \leq \sigma_0 \leq 0.9723$,

$$(3.7) \quad E_1(\sigma_0, T, H) \leq \frac{4H_0}{H_0-H} \left(\frac{(\log H_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)} \frac{\log H_0}{H_0} \right) \\ + \frac{\max\left(0, \frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2}\right)}{H_0} + \frac{(2-\sigma_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)^2} - \frac{\sigma_0 H_0^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{H_0^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{H_0^{-2\sigma_0-1}}{2}.$$

Theorem 1.2 gives

(3.8)

$$E_2(\sigma_0, T, H) \leq c_0^2 \frac{1}{T-H} \int_H^T t^{-2\sigma_0} dt \leq \frac{c_0^2}{2\sigma_0-1} \frac{H^{-(2\sigma_0-1)} - H_0^{-(2\sigma_0-1)}}{H_0 - H}.$$

We use the Cauchy-Schwarz inequality to bound E_3 :

$$\begin{aligned} E_3(\sigma_0, T, H) &\leq 2 \left(\frac{1}{T-H} \int_H^T |\Re R(s)|^2 dt \right)^{\frac{1}{2}} \left(\frac{1}{T-H} \int_H^T \left| \sum_{1 \leq n < t} \frac{1}{n^{\sigma_0+it}} \right|^2 dt \right)^{\frac{1}{2}} \\ &\leq 2 \sqrt{E_2(\sigma_0, T, H) (D(\sigma_0, T, H) + E_1(\sigma_0, T, H))} \\ (3.9) \quad &\leq 2 \sqrt{\epsilon_2(\sigma_0, H) (\zeta(2\sigma_0) + \epsilon_1(\sigma_0, H))}. \end{aligned}$$

The definitions of $\epsilon_1, \epsilon_2, \epsilon_3$ follow from (3.7), (3.8), and (3.9). The proof is achieved by putting together (3.1), (3.3), (3.7), (3.8), (3.9), and by applying the following bound for concave functions

$$\int_H^T \log |\zeta(\sigma_0 + it)| dt \leq \frac{T-H}{2} \log \left(\frac{1}{T-H} \int_H^T |\zeta(\sigma_0 + it)|^2 dt \right).$$

4. A LOWER BOUND FOR $\log |\zeta(s)|$ WHEN $\sigma > 1$.

Lemma 4.1. *Let $2 \leq H \leq T$ and $\sigma_1 = 1.5002$. Then*

$$(4.1) \quad \int_H^T \log |\zeta(\sigma_1 + it)| dt \geq -\mathcal{E}_2, \text{ with } \mathcal{E}_2 = 1.7655.$$

Proof. Let $s = \sigma_1 + it$. It follows from the Euler product that

$$\log |\zeta(s)| = \Re \sum_{n \geq 2} \frac{\Lambda(n)}{(\log n) n^s}.$$

Thus

$$\int_H^T \log |\zeta(\sigma_1 + it)| dt = \sum_{n \geq 2} \frac{\Lambda(n) (\sin(T \log n) - \sin(H \log n))}{(\log n)^2 n^{\sigma_1}} \geq -2 \sum_{n \geq 2} \frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}}.$$

We truncate the sum at $N_0 = 10^3$ and bound the tail

$$\sum_{n > N_0} \frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}} \leq \frac{1}{(\log N_0)^2} \left(-\frac{\zeta'}{\zeta}(\sigma_1) - \sum_{n \leq N_0} \frac{\Lambda(n)}{n^{\sigma_1}} \right).$$

We obtain

$$\int_H^T \log |\zeta(\sigma_1 + it)| dt \geq -2 \left(\frac{-\frac{\zeta'}{\zeta}(\sigma_1)}{(\log N_0)^2} + \sum_{n \leq N_0} \frac{\Lambda(n)}{n^{\sigma_1}} \left(\frac{1}{(\log n)^2} - \frac{1}{(\log N_0)^2} \right) \right),$$

and a numerical calculation with Maple gives the value for the above left term. \square

5. EXPLICIT BOUNDS FOR $\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau$.

Lemma 5.1. *Let $\eta = 0.0001, \sigma_1 = 3/2 + 2\eta = 1.5002$. Let σ_0, T, H satisfy $\sigma_0 < \sigma_1, 2 \leq H \leq T$. Then*

$$\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau \leq \mathcal{E}_3(\sigma_0) \log(HT) + \mathcal{E}_4(\sigma_0, H)$$

with

$$(5.1)$$

$$\mathcal{E}_3(\sigma_0) = \frac{\pi(1 + 2\eta)(\sigma_1 - \sigma_0)}{4 \log 2},$$

$$(5.2)$$

$$\mathcal{E}_4(\sigma_0, H) = \frac{\pi(\sigma_1 - \sigma_0)}{\log 2} \log \left(3 \frac{H + 3(1 + \eta)}{H - (1 + 2\eta)} \left(\frac{3(1 + \eta)/H + 1}{2\pi} \right)^{\frac{1+2\eta}{2}} \frac{\zeta(1 + \eta)^4}{\zeta(2(1 + \eta))^2} \right).$$

It suffices to bound an integral of the form

$$\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it) d\tau,$$

with $t \geq H$. We only make use of the convexity bound for $\zeta(s)$.

Proof. Let $\omega \in \mathbb{C}$ and $N \in \mathbb{N}$. Following Rosser's modification of Backlund's trick ([1, equation (32)] and [24, page 223]), we introduce $f_t(\omega) = \frac{1}{2} \left(\zeta(\omega + it)^N + \zeta(\omega - it)^N \right)$. We denote n to be the number of real zeros of $f_t(\tau) = \Re \zeta(\tau + it)^N$ in the interval $\sigma_0 < \tau < \sigma_1$. The interval is split into $n + 1$ subintervals and on each of them $\arg \zeta(\tau + it)^N$ changes by at most π . Thus

$$(5.3)$$

$$\left| \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it) d\tau \right| = \frac{1}{N} \left| \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it)^N d\tau \right| \leq \frac{(\sigma_1 - \sigma_0)(n + 1)\pi}{N}.$$

We denote $n(r)$ the number of zeros of f_t in the circle centered at $1 + \eta + it$, and with radius r . For $r \geq 1/2 + \eta$, the segment $[\sigma_0, \sigma_1]$ is contained in $[1 + \eta - r, 1 + \eta + r]$, thus $n \leq n(r)$. The following version of Jensen's formula [29, p. 137, equation (2)],

$$\log |f_t(1 + \eta)| + \int_0^{1+2\eta} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \log |f_t(1 + \eta + (1 + 2\eta)e^{i\theta})| d\theta,$$

allows us to deduce an upper bound for n :

$$(5.4)$$

$$n \leq \frac{1}{\log 2} \int_0^{1+2\eta} \frac{n(r)}{r} dr \leq \frac{1}{2\pi \log 2} \int_{-\pi/2}^{3\pi/2} \log |f_t(1 + \eta + (1 + 2\eta)e^{i\theta})| d\theta - \frac{\log |f_t(1 + \eta)|}{\log 2}.$$

We write $\zeta(1 + \eta + it) = Re^{i\phi}$. Thus $f_t(1 + \eta) = \Re \left(\zeta(1 + \eta + it)^N \right) = R^N \cos(N\phi)$. We choose a sequence of N 's such that $\lim_{N \rightarrow \infty} N\phi = 0 \pmod{2\pi}$.

Thus

$$(5.5) \quad \begin{aligned} \log |f_t(1+\eta)| &= N \log((1+o(1))R) = N \log((1+o(1))|\zeta(1+\eta+it)|) \\ &\geq N \log\left(\frac{\zeta(2(1+\eta))}{\zeta(1+\eta)}\right) + o_N(1), \end{aligned}$$

where $o_N(1) \rightarrow 0$ when $N \rightarrow \infty$. We now split the integral in the left term of inequality (5.4) depending on the sign of $\cos \theta$. For $\theta \in (-\pi/2, \pi/2)$, $\Re(1+\eta+(1+2\eta)e^{i\theta} \pm it) > 1+\eta > 1$, and we use the trivial bound

$$|\zeta(1+\eta+(1+2\eta)e^{i\theta} \pm it)| \leq \zeta(1+\eta),$$

giving

$$(5.6) \quad \int_{-\pi/2}^{\pi/2} \log |f_t(1+\eta+(1+2\eta)e^{i\theta})| d\theta \leq N\pi \log(\zeta(1+\eta)).$$

For $\theta \in (\pi/2, 3\pi/2)$, we use Rademacher's bound [17, equation (7.4)]:

$$|\zeta(s)| \leq 3 \frac{|1+s|}{|1-s|} \left(\frac{|1+s|}{2\pi} \right)^{\frac{1+\eta-\Re s}{2}} \zeta(1+\eta)$$

with $s = 1+\eta+(1+2\eta)e^{i\theta} \pm it$. Since

$$|1+s| \leq t+3(1+\eta), \quad |1-s| \geq |\Im s| \geq t-(1+2\eta), \quad \text{and } 0 \leq 1+\eta-\Re s \leq 1+2\eta,$$

then

$$(5.7) \quad \begin{aligned} \int_{\pi/2}^{3\pi/2} \log |f_t(1+\eta+(1+2\eta)e^{i\theta})| d\theta \\ \leq N\pi \log \left(3 \frac{t+3(1+\eta)}{t-(1+2\eta)} \left(\frac{t+3(1+\eta)}{2\pi} \right)^{\frac{1+2\eta}{2}} \zeta(1+\eta) \right). \end{aligned}$$

Together with (5.4), (5.5), (5.6), and (5.7), we deduce

$$(5.8) \quad \begin{aligned} n &\leq \frac{N}{2\log 2} \log \left(3 \frac{t+3(1+\eta)}{t-(1+2\eta)} \left(\frac{3(1+\eta)+t}{2\pi} \right)^{\frac{1+2\eta}{2}} \frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2} \right) + o_N(1) \\ &\leq \frac{N(1+2\eta)}{4\log 2} \log t + \frac{N}{2\log 2} \log \left(3 \frac{t+3(1+\eta)}{t-(1+2\eta)} \left(\frac{3(1+\eta)/t+1}{2\pi} \right)^{\frac{1+2\eta}{2}} \frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2} \right) + o_N(1). \end{aligned}$$

Together with (5.3) and letting $N \rightarrow \infty$, we obtain

$$\begin{aligned} \left| \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau+it) d\tau \right| &\leq \frac{\pi(1+2\eta)(\sigma_1-\sigma_0)}{4\log 2} \log t \\ &\quad + \frac{\pi(\sigma_1-\sigma_0)}{2\log 2} \log \left(3 \frac{t+3(1+\eta)}{t-(1+2\eta)} \left(\frac{3(1+\eta)/t+1}{2\pi} \right)^{\frac{1+2\eta}{2}} \frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2} \right). \end{aligned}$$

Observing that the second term decreases with $t \geq H$ achieves the proof. \square

6. EXPLICIT UPPER BOUNDS FOR $N(\sigma, T)$ - PROOF OF THEOREM 1.1

Proof. We recall that $\sigma, \sigma_0, \sigma_1, H$ and T satisfy (1.11). We consider the number $N(\sigma, T)$ of zeros $\varrho = \beta + i\gamma$ of zeta in the rectangle $\sigma < \beta < 1$ and $H < \gamma < T$. Since $N(\sigma, H) = 0$, we have

$$(6.1) \quad N(\sigma, T) \leq \frac{1}{\sigma - \sigma_0} \int_{\sigma_0}^{\sigma_1} \left(N(\tau, T) - N(\tau, H) \right) d\tau.$$

It follows from a lemma of Littlewood (see [31, (9.9.1)]) that

$$\int_{\sigma_0}^{\sigma_1} \left(N(\tau, T) - N(\tau, H) \right) d\tau = -\frac{1}{2\pi i} \int_{\mathcal{R}} \log \zeta(s) ds,$$

where \mathcal{R} is the rectangle with vertices $\sigma_0 + iH$, $\sigma_1 + iH$, $\sigma_1 + iT$, and $\sigma_0 + iT$.

Thus

$$(6.2) \quad N(\sigma, T) \leq \frac{1}{2\pi(\sigma - \sigma_0)} \left(\int_H^T \log |\zeta(\sigma_0 + it)| dt + \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau \right. \\ \left. - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau - \int_H^T \log |\zeta(\sigma_1 + it)| dt \right).$$

We use Theorem 1.4, Lemma 4.1, and Lemma 5.1 respectively to bound these integrals:

$$\int_H^T \log |\zeta(\sigma_0 + it)| dt \leq \frac{T - H}{2} \log \left(\zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right), \\ - \int_H^T \log |\zeta(\sigma_1 + it)| dt \leq \mathcal{E}_2, \\ \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau \leq \mathcal{E}_3(\sigma_0) \log(HT) + \mathcal{E}_4(\sigma_0, H),$$

where the \mathcal{E}_i 's are defined respectively in (1.10), (4.1), (5.1), and (5.2). We obtain

$$N(\sigma, T) \leq b_1(\sigma_0, H)(T - H) + b_2(\sigma_0, H) \log(TH) + b_3(\sigma_0, H),$$

with

(6.3)

$$b_1(\sigma_0, H) = \frac{\log \left(\zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right)}{4\pi(\sigma - \sigma_0)}, \quad b_2(\sigma_0, H) = \frac{\mathcal{E}_3(\sigma_0)}{2\pi(\sigma - \sigma_0)}, \quad b_3(\sigma_0, H) = \frac{\mathcal{E}_2 + \mathcal{E}_4(\sigma_0, H)}{2\pi(\sigma - \sigma_0)}.$$

It follows

$$N(\sigma, T) \leq c_1 T + c_2 \log T + c_3, \quad \text{with } c_1 = b_1, \quad c_2 = b_2, \quad c_3 = -b_1 H + b_2 \log H + b_3.$$

□

Table 1 records values of the b_i 's and c_i 's computed for $H_0 = 3.061 \cdot 10^{10}$. Specific choices of parameters σ_0 and H are chosen in order to obtain good bounds for $N(\sigma, T)$ when T is asymptotically large. The values of σ_0, b_1, c_1, b_2 , and c_2 displayed in the table are rounded up to 4 decimal places. We take the ceiling of the values of H, b_3 , and c_3 .

TABLE 1. $N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3$ and $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$, for $T \geq H_0$.

σ	σ_0	H	$b_1 = c_1$	$b_2 = c_2$	b_3	c_3
0.60	0.5229	19 399	4.2288	2.2841	333	-81 673
0.65	0.5552	40 105	2.4361	1.7965	262	-97 414
0.70	0.5873	91 470	1.4934	1.4609	213	-136 370
0.75	0.6096	169 119	1.0031	1.1442	167	-169 449
0.76	0.6136	188 973	0.9355	1.0921	160	-176 604
0.77	0.6175	210 645	0.8750	1.0437	153	-184 134
0.78	0.6213	234 346	0.8205	0.9986	146	-192 120
0.79	0.6250	260 321	0.7714	0.9566	140	-200 644
0.80	0.6287	288 853	0.7269	0.9176	134	-209 795
0.81	0.6324	320 270	0.6864	0.8812	129	-219 667
0.82	0.6361	354 951	0.6495	0.8473	124	-230 367
0.83	0.6398	393 341	0.6156	0.8157	119	-242 009
0.84	0.6435	435 955	0.5846	0.7862	115	-254 724
0.85	0.6472	483 393	0.5561	0.7586	111	-268 658
0.86	0.6510	536 357	0.5297	0.7327	107	-283 978
0.87	0.6548	595 670	0.5053	0.7085	104	-300 872
0.88	0.6587	662 291	0.4827	0.6857	101	-319 555
0.89	0.6626	737 343	0.4617	0.6644	97	-340 272
0.90	0.6667	822 142	0.4421	0.6443	95	-363 301
0.91	0.6708	918 225	0.4238	0.6253	92	-388 959
0.92	0.6750	1 027 390	0.4066	0.6075	89	-417 606
0.93	0.6793	1 151 729	0.3905	0.5906	87	-449 647
0.94	0.6838	1 293 683	0.3754	0.5747	84	-485 543
0.95	0.6883	1 456 079	0.3612	0.5596	82	-525 807
0.96	0.6930	1 642 194	0.3478	0.5452	80	-571 018
0.97	0.6977	1 855 803	0.3352	0.5316	78	-621 815
0.98	0.7026	2 101 249	0.3232	0.5187	76	-678 911
0.99	0.7077	2 383 498	0.3118	0.5063	74	-743 087

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REFERENCES

- [1] R. J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Mathematica, vol. 41 (1917), 345–375.
- [2] Y. Cheng, *An explicit upper bound for the Riemann Zeta-function near the line $\sigma = 1$* , Rocky Mountain J. Math. 29, Number 1 (1999), 115–140.
- [3] Y. Cheng, *Explicit estimate on primes between consecutive cubes*, Rocky Mountain J. Math. 40, Number 1 (2010), 117–153.
- [4] H. Daboussi and J. Rivat, *Explicit upper bounds for exponential sums over primes*, Math. Comp. 70 (2001), no. 233, 431–447.
- [5] P. Dusart, *Autour de la fonction qui compte le nombre de nombres premiers* Thèse, Université de Limoges, 1998.
- [6] P. Dusart, *Inégalités explicites pour $\psi(x)$, $\theta(x)$, $\pi(x)$ et les nombres premiers*, C. R. Math. Acad. Sci. Soc. R. Can. 21 (1999), no. 2, 53–59.
- [7] P. Dusart, *The k^{th} prime is greater than $k(\log k + \log \log k - 1)$ for $k \geq 2$* , Math. Comp. 68 (1999), no. 225, 411–415.
- [8] P. Dusart, *Estimates of some functions over primes without RH*, arXiv preprint arXiv:1002.0442, 2010.
- [9] L. Faber and H. Kadiri, *New bounds for $\psi(x)$* , preprint.
- [10] K. Ford, *Zero-free regions for the Riemann zeta function*, Number theory for the millennium, II (Urbana, IL, 2000), 25–56.
- [11] X. Gourdon, *The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height*, available at <http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf>.
- [12] G.H. Hardy & J.E. Littlewood, *The zeros of the Riemann zeta-function on the critical line*, Math. Z. 10 (1921), 283–317.
- [13] H. Helfgott, *Minor arcs for Goldbach problem*, preprint, arXiv:1205.5252.
- [14] H. Kadiri *Une région explicite sans zéros pour la fonction ζ de Riemann*, Acta Arith. 117 (2005), no. 4, 303–339.
- [15] D. Platt, *Computing degree 1 L-functions rigorously*, Ph.D. Thesis, University of Bristol, 2011.
- [16] D. Platt, *Computing zeta on the half-line*, preprint.
- [17] H. Rademacher, *On the Phragmén-Lindelöf theorem and some applications*, Math. Z. 72 (1959), 192–204.
- [18] O. Ramaré, *On Snirelman’s constant*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995), no. 4, 645–706.
- [19] O. Ramaré, *An explicit density estimate*, preprint.
- [20] O. Ramaré, *Explicit estimates for the summatory function of $\Lambda(n)/n$ from the one of $\Lambda(n)$* , to appear in Acta Arith.
- [21] O. Ramaré, *Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions*, preprint, available at <http://math.univ-lille1.fr/~ramare/Maths/contenu.html>.
- [22] O. Ramaré, *From explicit estimates for the primes to explicit estimates for the Moebius function*, to appear in Acta Arith.
- [23] O. Ramaré and R. Rumely, *Primes in Arithmetic Progressions*, Math. Comp. 65 (1996), no. 213, 397–425.
- [24] J.B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. 63 (1941), 211–232.
- [25] J.B. Rosser and L. Schoenfeld *Approximate formulas for some functions of prime numbers*, Illinois. J. Math. 6 (1962), 64–94.

- [26] J.B. Rosser and L. Schoenfeld *Sharper bounds for Chebyshev functions $\psi(x)$ and $\theta(x)$* , Math. Comp. 29 (1975), 243–269.
- [27] J.B. Rosser and L. Schoenfeld *Sharper bounds for Chebyshev functions $\psi(x)$ and $\theta(x)$ II*, Math. Comp. 30 (1976), no. 134, 337–360.
- [28] L. Schoenfeld, *Sharper bounds for the Chebyshev functions ψ and θ II*, Math. Comp. 30 (1976), 337–360.
- [29] E.M. Stein & R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis II, Princeton University Press.
- [30] T. Tao, *Every odd number greater than 1 is the sum of at most five primes*, to appear in Math. Comp.
- [31] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, second edition, Oxford Science Publications.
- [32] E.C. Titchmarsh, *The Theory of Functions*, second edition, Oxford University Press.
- [33] T. Trudgian, *An improved upper bound for the argument of the Riemann zeta-function on the critical line II*, preprint, arXiv:1208.5846.
- [34] J. van de Lune, H.J.J. te Riele, D.T. Winter *On the zeros of the Riemann zeta-function in the critical strip. IV* Math. Comp. 46 (1986), no. 174, 667–681.
- [35] S. Wedeniwski - Zetagrid, *Computational verification of the Riemann hypothesis* Conference in Number Theory in Honour of Professor H.C. Williams, Alberta, Canada, May 2003.
<http://www.zetagrid.net/zeta/math/zeta.result.100billion.zeros.html>

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